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Resummations with renormalon effects for the hadronic vacuum polarization contribution to the muon (g-2)

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Abstract

The hadronic vacuum polarization contribution to the muon $(g-2)$ value is calculated by considering a known dispersion integral which involves the $R_{e^+e^-}(s)$ ratio. The theoretical part stemming from the region below 1.8 GeV is the largest contribution in our approach, and is calculated by using a contour integral involving the associated Adler function $D(Q^2)$. In the resummations, we explicitly take into account the exactly known renormalon singularity of the leading infrared renormalon in the usual and in the modified Borel transform of $D(Q^2)$, and map further away from the origin the other renormalon singularities by employing judiciously chosen conformal transformations. The renormalon effect increases the predicted value of the hadronic vacuum polarization contribution to the muon $(g-2)$, and therefore diminishes the difference between the recently measured and the SM/QCD-predicted value of $(g-2)$. It is also shown that the total QED correction to the hadronic vacuum polarization is very small, about 0.06 %.

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I. INTRODUCTION

The new precise measurement of the muon anomalous magnetic moment $a_\mu \equiv (g-2)/2$ [1] allows for detailed testing of the standard model, and therefore for the possibility of looking into physics beyond the standard model as well. In fact, comparison of the experimental result with some theoretical calculations shows a 2.6σ difference [1]. This has been suggested as the appearance of new physics.[1] Since the advertised discrepancy comes from the calculation of Ref. [8] of the hadronic vacuum polarization contributions $a_\mu^{(\text{v.p.})}$ to a_μ, we re–evaluate in the present paper the theoretical (pQCD+OPE) parts of this quantity. We use the same theoretical approach as in Ref. [8], which in turn is based on the approach of Ref. [9]. However, in addition, we take into account the known renormalon structure of the Adler function.

II. FORMALISM

According to Ref. [10], the hadronic vacuum polarization contribution to the muon anomalous magnetic moment $a_\mu \equiv (g-2)/2$ appears in the following dispersion integral:

$$a_\mu^{(\text{v.p.})} = \frac{\alpha_{em}^2(0)}{3\pi^2} \int_{4m_\pi^2}^{\infty} \frac{ds}{s} K(s) R_{e^+e^-}(s) \ , \tag{1}$$

where $K(s)$ is the QED kernel [10]

$$K(s) = x^2\left(1-\frac{x^2}{2}\right) + (1+x)^2\left(1+\frac{x^2}{2}\right)\left[\ln(1+x) - x + \frac{x^2}{2}\right] + \frac{(1+x)}{(1-x)} x^2 \ln x \ . \tag{2}$$

Here, $x = (1-y_\mu)/(1+y_\mu)$ with $y_\mu = (1-4m_\mu^2/s)^{1/2}$. The largest part (about 92%) of $a_\mu^{(\text{v.p.})}$ comes from the region with CMS energy $\sqrt{s} < \sqrt{s_0} = 1.8$ GeV. Following the approach of Ref. [9], applied in Ref. [8] to $a_\mu^{(\text{v.p.})}$, we rewrite the dispersion integral (1) (with $s_{\max} = s_0$) in the form

$$\begin{aligned} \frac{3\pi^2}{\alpha_{em}^2(0)} \times a_\mu^{(\text{v.p.})}(s \le s_0) = & \int_{4m_\pi^2}^{s_0} ds\ R_{e^+e^-}(s) \left[\frac{K(s)}{s} - C_1\left(1-\frac{s}{s_0}\right)\right] \\ & + C_1 \int_{4m_\pi^2}^{s_0} ds\ R_{e^+e^-}(s)\left(1-\frac{s}{s_0}\right) \ . \end{aligned} \tag{3}$$

Here C_1 is in principle an arbitrary constant, which however, according to the philosophy of Ref. [9], may be chosen in such a way as to minimize the first ("data") integral and maximize the second ("theory") integral. It is known that $R_{e^+e^-}(s) = 12\pi \text{Im}\Pi(s+i\varepsilon)$, with $\Pi(s)$ being the hadronic part of the (vector) photon vacuum polarization function which has no poles

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[1] For an (incomplete) list of works investigating the possibility to explain and/or to find implications of this difference within various frameworks of new physics, see: Refs. [2] (supersymmetric models), [3] (non-supersymmetric extended Higgs sectors), [4] (composite models), [5] (models with extra dimensions), [6] (leptoquarks), and [7] (other models).

in the interval $[0, 4m_\pi^2)$; further, $(1 - s/s_0)$ also has no poles in that interval, in contrast to the function $K(s)$. Therefore, the Cauchy theorem can be applied to the second ("theory") integral, with the path of Fig. 1. Carrying subsequently integration by parts, and using the identity $D(Q^2 \equiv -s) = -12\pi^2 s \, d\Pi(s)/ds$, leads to

$$\frac{3\pi^2}{\alpha_{em}^2(0)} \times a_\mu^{(\text{v.p.})}(s \le s_0) = \int_{4m_\pi^2}^{s_0} ds \, R_{e^+e^-}(s) \left[\frac{K(s)}{s} - C_1 \left(1 - \frac{s}{s_0} \right) \right] + \frac{C_1 s_0}{4\pi} \int_{-\pi}^{\pi} dy \, D(Q^2 = s_0 e^{iy})(1 + e^{iy})^2 \ , \qquad (4)$$

where the associated vector Adler function $D(Q^2)$ can be written in the following way:[2]

$$D(Q^2) = N_c \sum_{f=u,d,s} Q_f^2 \left[1 + D_{\text{can.}}(Q^2) + 4\pi^2 \times \sum_{d=2,4,\ldots} D_{f\bar{f};V}^{(d;J=1)}(Q^2) \right] \ . \qquad (5)$$

Here, $D_{\text{can.}}(Q^2)$ is the canonically normalized massless QCD part with dimension $d = 0$, whose power expansion in $a^{\overline{\text{MS}}}(Q^2) \equiv \alpha_s^{\overline{\text{MS}}}(Q^2)/\pi$ is

$$D_{\text{can.}}(Q^2) = a^{\overline{\text{MS}}}(Q^2) \times \left[1 + d_1^{(0)} a^{\overline{\text{MS}}}(Q^2) + d_2^{(0)} \left(a^{\overline{\text{MS}}}(Q^2) \right)^2 + d_3^{(0)} \left(a^{\overline{\text{MS}}}(Q^2) \right)^3 + \cdots \right] \ , \qquad (6)$$

with $d_1^{(0)} = 1.6398$ [11], $d_2^{(0)} = 6.3710$ [12], and $d_3^{(0)}$ is estimated to be $d_3^{(0)} = 25 \pm 10$ [13]. The renormalization scale in (6) is $\mu^2 = Q^2$, and the renormalization scheme is $\overline{\text{MS}}$. The number of active quark flavors is $n_f = 3$. The $d = 2$ contributions are [14,15]

$$4\pi^2 D_{f\bar{f};V}^{(d=2;J=1)}(Q^2) = -\frac{6 m_f^2(Q^2)}{Q^2} \left[1 + \frac{14}{3} a^{\overline{\text{MS}}}(Q^2) + 43.0581 \left(a^{\overline{\text{MS}}}(Q^2) \right)^2 + \mathcal{O}(a^3) \right] \ , \qquad (7)$$

where only the s quark contributes appreciably. The $d = 4$ contributions are those of the gluon condensate [14]

$$4\pi^2 D_{f\bar{f};V}^{(\text{glc};\text{d}=4;\text{J}=1)}(Q^2) = +\frac{2\pi^2}{3(Q^2)^2} \langle aGG \rangle \left[1 - \frac{11}{18} a^{\overline{\text{MS}}}(Q^2) + \mathcal{O}(a^2) \right] \ , \qquad (8)$$

those of the quark mass condensates [14]

$$4\pi^2 D_{f\bar{f};V}^{(\text{qc.};\text{d}=4;\text{J}=1)}(Q^2) = \frac{16\pi^2}{(Q^2)^2} \left[1 + \frac{1}{3} a^{\overline{\text{MS}}}(Q^2) + \frac{11}{2} \left(a^{\overline{\text{MS}}}(Q^2) \right)^2 + \mathcal{O}(a^3) \right] \langle m_f \bar{f} f \rangle + \frac{8\pi^2}{(Q^2)^2} \left[\frac{4}{27} a^{\overline{\text{MS}}}(Q^2) + 1.074 \left(a^{\overline{\text{MS}}}(Q^2) \right)^2 + \mathcal{O}(a^3) \right] \sum_{f_k=u,d,s} \langle m_{f_k} \bar{f}_k f_k \rangle \ , \qquad (9)$$

and those proportional to m_f^4 [14]

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[2] This Adler function does not contain QED radiative corrections. The effects of them will be added later on.

$$4\pi^2 D^{(\mathrm{qm;d=4;J=1})}_{f\bar{f};V}(Q^2) = \frac{48}{7(Q^2)^2}\left[-\frac{1}{a^{\overline{\mathrm{MS}}}(Q^2)} + 1 + 12.0 a^{\overline{\mathrm{MS}}}(Q^2) + \mathcal{O}(a^2)\right] m_f^4(Q^2)$$
$$-\frac{2}{7(Q^2)^2}\left[1 + 8.4 a^{\overline{\mathrm{MS}}}(Q^2) + \mathcal{O}(a^2)\right] \sum_{f_k=u,d,s} m_{f_k}^4(Q^2) \ . \qquad (10)$$

The terms with dimension $d \geq 6$ do not contribute to the "theory" part in (4) in the leading order renormalization group (RG) approximation. We note that the leading term in (9) is twice as large as that in [8] [their Eq. (9)]. We will see later that the terms (10) give negligible contributions, but not the terms (7)–(9). In the quark condensate terms (9) we can use the (approximate) leading PCAC relations $m_s\langle\bar{q}q\rangle \approx -f_\pi^2(m_K^2 - m_\pi^2)$ and $(m_u + m_d)\langle\bar{q}q\rangle \approx -f_\pi^2 m_\pi^2$ ($q = u, d$ or s; $f_\pi = 0.0924 \pm 0.0003$ GeV)

$$\langle A^{(1)}_{\mathrm{qc}}\rangle \equiv \sum_{f=u,d,s} Q_f^2 \langle m_f \bar{f} f\rangle \approx -2.50 \times 10^{-4}\ \mathrm{GeV}^4 \ , \qquad (11)$$
$$\langle A^{(2)}_{\mathrm{qc}}\rangle \equiv \sum_{f=u,d,s} \langle m_f \bar{f} f\rangle \approx -2.08 \times 10^{-3}\ \mathrm{GeV}^4 \ . \qquad (12)$$

Within the leading PCAC approach, the uncertainties in the numbers in (11)–(12) originate from the uncertainty of f_π^2, and of the ratio $\epsilon_u \equiv m_u/m_s = 0.029 \pm 0.003$ [16,15]. They are $\sim 1\%$, and affect insignificanly the results of the present paper.

Insertion of the expressions (6)–(10) into the contour integral (4) gives

$$\frac{3\pi^2}{\alpha_{em}^2(0)} \times a_\mu^{(\mathrm{v.p.})}(s \leq s_0)^{(\mathrm{theor.})} = C_1 s_0 \times \Bigg\{ 1 + A_{\mathrm{can.}}$$
$$-\frac{1}{2\pi}\frac{m_s^2(s_0)}{s_0} M_{0,2}\left[1 + \frac{14}{3}\frac{M_{1,2}}{M_{0,2}} + 43.0581\frac{M_{2,2}}{M_{0,2}}\right] + \frac{\pi}{3}\frac{\langle aGG\rangle}{s_0^2} A_{0,4}\left[1 - \frac{11}{18}\frac{A_{1,4}}{A_{0,4}}\right]$$
$$+12\pi\frac{1}{s_0^2}\langle A^{(1)}_{\mathrm{qc}}\rangle A_{0,4}\left[1 + \frac{1}{3}\frac{A_{1,4}}{A_{0,4}} + \frac{11}{2}\frac{A_{2,4}}{A_{0,4}}\right] + 4\pi\frac{1}{s_0^2}\langle A^{(2)}_{\mathrm{qc}}\rangle \frac{4}{27} A_{1,4}\left[1 + 7.25\frac{A_{2,4}}{A_{1,4}}\right] L$$
$$-\frac{4}{7\pi}\frac{m_s^4(s_0)}{s_0^2} M_{-1,4}\left[1 - \frac{M_{0,4}}{M_{-1,4}} - 12.\frac{M_{1,4}}{M_{-1,4}}\right] - \frac{1}{7\pi}\frac{m_s^4(s_0)}{s_0^2} M_{0,4}\left[1 + 8.4\frac{M_{1,4}}{M_{0,4}}\right] \Bigg\} \ . \qquad (13)$$

Here, we used the complex momentum contour integrals

$$A_{\mathrm{can.}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} dy (1 + \mathrm{e}^{\mathrm{i}y})^2 D_{\mathrm{can.}}(Q^2 = s_0 \mathrm{e}^{\mathrm{i}y}) \ , \qquad (14)$$
$$A_{n,2k} = \int_{-\pi}^{\pi} dy (1 + \mathrm{e}^{\mathrm{i}y})^2 \mathrm{e}^{-\mathrm{i}ky} \left(a^{\overline{\mathrm{MS}}}(Q^2 = s_0 \mathrm{e}^{\mathrm{i}y})\right)^n \ , \qquad (15)$$
$$M_{n,2k} = \int_{-\pi}^{\pi} dy (1 + \mathrm{e}^{\mathrm{i}y})^2 \mathrm{e}^{-\mathrm{i}ky} \left(\frac{m_s(s_0 \mathrm{e}^{\mathrm{i}y})}{m_s(s_0)}\right)^{2k} \left(a^{\overline{\mathrm{MS}}}(Q^2 = s_0 \mathrm{e}^{\mathrm{i}y})\right)^n \ . \qquad (16)$$

For RGE evolution of $a^{\overline{\mathrm{MS}}}(Q^2)$ we use the four–loop truncated perturbation series (TPS) [17] of the $\overline{\mathrm{MS}}$ beta function, with $n_f = 3$. In addition, for the RGE evolution of $m_s(Q^2)$ we use the $\overline{\mathrm{MS}}$ four–loop TPS quark mass anomalous dimension [18]. For example, the RGE evolution along the complex momentum contour gives

$$\frac{m_s(s_0 \mathrm{e}^{\mathrm{i}y})}{m_s(s_0)} = \exp\left\{ -\mathrm{i}\int_0^y dy' a^{\overline{\mathrm{MS}}}(s_0 \mathrm{e}^{\mathrm{i}y'}) \left[1 + \sum_{n=1}^{3} \widetilde{\gamma}_n \left(a^{\overline{\mathrm{MS}}}(s_0 \mathrm{e}^{\mathrm{i}y'})\right)^n\right] \right\} , \qquad (17)$$

with $\widetilde{\gamma}_1 = 3.79167$, $\widetilde{\gamma}_2 = 12.4202$, and $\widetilde{\gamma}_3 = 44.263$ [18].

III. EVALUATION

We first use the input values as used in Ref. [8]

$$\langle aGG \rangle = (0.015 \pm 0.020) \ \mathrm{GeV}^4, \tag{18}$$

$$\alpha_s^{\overline{\mathrm{MS}}}(m_\tau^2) = 0.333 \pm 0.017 \quad \left(\Rightarrow \ \alpha_s^{\overline{\mathrm{MS}}}(M_z^2) \approx 0.1201 \pm 0.0020 \right) \ , \tag{19}$$

$$m_s(1\mathrm{GeV}^2) = 0.20 \pm 0.07 \ \mathrm{GeV} \quad \Rightarrow \ m_s(m_\tau^2) \approx 0.151 \pm 0.053 \ \mathrm{GeV} \ . \tag{20}$$

Further, we use for $D_{\mathrm{can.}}(Q^2)$ the NNLO TPS (i.e., with $d_3^{(0)} = 0$) and with the renormalization scale $\mu^2 = Q^2$, i.e., the approach apparently used by [8]. The result for their input values, and for $C_1 = 0.007 \ \mathrm{GeV}^{-2}$, is then:

$$10^{11} \times a_\mu^{(\mathrm{v.p.})}(s \leq s_0; \mathrm{th.\ part}; \mathrm{NNLO}\ D_{\mathrm{can.}}) = 4752 \pm 108 \ , \tag{21}$$

in contrast to their value (4686.2 ± 113.2). The central values of these theory parts are thus higher by 1.4% than those given in [8]. This percentage does not change at different values of the parameter C_1, since the results are linearly proportional to C_1. The uncertainty ± 108 in (21) is obtained by adding in quadrature the uncertainty from α_s (± 61), from m_s (± 74), and from $\langle aGG \rangle$ (± 49). The separate contributions to the central value 4752 in (21) are: 4079 from the leading term; 749 from the (resummed) canonical part (6) [$\Rightarrow$ (14)]; -87 from the $d=2$ strange mass term (7); 37 from the $d=4$ gluon condensate term (8); -25 from the $d=4$ quark condensate terms (9); -1 from the $d=4$ quark mass terms (10).

We can, however, re–calculate the canonical part (14) by using methods which account for the renormalon structure of the Adler function. The latter structure has been shown [13,19] to have numerically significant effects in the hadronic τ decay width ratio.[3] Ref. [13] uses ordinary Borel transforms, and Ref. [19] modified Borel transforms of $D_{\mathrm{can.}}$. In these two References, the exactly known leading infrared renormalon singularity of the Borel transform $\widetilde{D}_{\mathrm{can.}}(b)$, at the value of the Borel variable $b=2$, has been fully taken into account as the pre–factor function of the form $1/(1-b/2)^{1+\nu}$ and $1/(1-b/2)$, respectively. This procedure allows us to describe the Borel transform more accurately in the most important region of the Borel integration, namely, the interval between the origin and the leading IR renormalon at $b = 2$. Furthermore, judiciously chosen conformal transformations $b = b(w)$ have been employed there to map the singularities of other renormalons further away from the origin and thus to reduce their numerical significance. For details of the calculation procedures, we refer to these two References.

If we re–calculate the caconical part (14) by using the ordinary Borel transform method of Ref. [13], the result (21) increases further to 4817 ± 116, as a consequence of the renormalon

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[3] In Refs. [13,19], the massless QCD quantity r_τ was calculated (resummed). It is obtained from the hadronic τ decay width ratio R_τ by subtracting from it the strangeness–changing contributions, factoring out the CKM–element $|V_{ud}|^2$ and the electroweak correction factors, and subtracting the quark mass ($m_{u,d} \neq 0$) contributions. The quantity r_τ is the same kind of contour integral in the complex momentum plane as the quantity $A_{\mathrm{can.}}$ of Eq. (14), but with the contour factor $(1+\mathrm{e}^{\mathrm{i}y})^2$ replaced by $(1+\mathrm{e}^{\mathrm{i}y})^3(1-\mathrm{e}^{\mathrm{i}y})$. Therefore, the analyses of Refs. [13,19] can be repeated, without any major changes, for the resummation of the quantity $A_{\mathrm{can.}}$ of Eq. (14).

effects. In $D_{\rm can.}$ we took $d_3^{(0)} = 25$, and for the renormalization scale (RScl) in $D_{\rm can.}$ we took the value which gives us the local insensitivity of the result with respect to RScl ($\mu^2 \approx 1.6 Q^2$), as argued in [13].

If we apply to expression (14) the method of resummation of Ref. [19] which employs modified Borel transforms, with the same input and with $d_3^{(0)} = 25 \pm 10$,[4] we obtain a value very similar to the aforementioned one

$$10^{11} \times a_\mu^{(\rm v.p.)}(s \leq s_0; \text{th. part}; \text{meth. [19]}) = 4820 \pm 120 \ . \tag{22}$$

The uncertainty ± 120 in (22) is obtained by adding in quadrature the uncertainty from α_s (± 74), from m_s (± 75), from $\langle aGG \rangle$ (± 49), as well as from the resummation method uncertainty and the uncertainty of $d_3^{(0)}$ (± 28). This calculation and the uncertainty estimates procedure are carried out in complete analogy with Ref. [19], to which we refer for details. The separate contributions to the central value 4820 in (22) are the same as those to (21), except that the contribution from the (resummed) canonical part (6) is now 817 (before: 749).

In order to isolate the contribution of the renormalon structure included in the value (22), we should compare the latter central value with the one obtained by using for $D_{\rm can}$ the N^{3}LO TPS with $d_3^{(0)} = 25$. The central value result in this case is 4772. The obtained renormalon structure effect is thus 48 units, or 1.0%.

The input values (19)–(20), used by the authors of Ref. [8], and taken up until now in the present work, can be replaced by what we believe to be more updated values

$$\alpha_s^{\overline{\rm MS}}(m_\tau^2) = 0.3254 \pm 0.0124 \quad \left(\Rightarrow \ \alpha_s^{\overline{\rm MS}}(M_z^2) \approx 0.1192 \pm 0.0015\right) \ , \tag{23}$$

$$m_s(m_\tau^2) = 0.119 \pm 0.024 \text{ GeV} \ . \tag{24}$$

The values (23) were obtained in [19] by a detailed analysis of the R_τ ratio, involving modified Borel transforms, and accounting for the renormalon structure of the associated Adler function via an explicit ansatz and with judiciously chosen conformal transformations. This result virtually agrees with the one obtained in the R_τ–analysis of Ref. [13] where ordinary Borel transforms were used instead. The values (23) are shifted downwards and the uncertainties are reduced, in comparison to the values (19). The latter values are based largely on the ALEPH analysis of the R_τ decay [20]. The latter analysis did not account for the renormalon structure of the associated Adler function. The trend towards smaller values of α_s and towards smaller uncertainties appears also in the analysis of the R_τ ratio of the authors of Ref. [21], who accounted for the renormalon structure via a large–β_0 resummation of the ordinary Borel transform and employed a resummation related to the effective charge (ECH) method – they obtained $\alpha_s(m_\tau^2) = 0.330 \pm 0.014$. The values (24) for the strange quark mass, which are significantly lower than those in (20), were obtained in the recent analysis of Ref. [15].

Further, ALEPH analysis [20] of the τ decays predicts the gluon condensate term to be consistent with zero

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[4] We refer to [13] for a detailed discussion of this estimate of $d_3^{(0)}$ values.

$$\langle aGG \rangle = (0.001 \pm 0.015)\ \text{GeV}^4\ , \tag{25}$$

in contrast with the input (18). We will take, in addition to the (α_s, m_s)–inputs (23)–(24), either the input (18) or (25) for the gluon condensate term. Small values of the gluon condensate close to the ALEPH values (25) are also suggested in the formalism of Ref. [22], where the power–suppressed terms are obtained from the knowledge of the perturbation series of D_{can} (6) and of its infrared renormalon structure.

Applying then again the resummation method of Ref. [19] to expression (14), we obtain the prediction

$$\begin{aligned} 10^{11} \times a_\mu^{(\text{v.p.})}(s \leq s_0; \text{th. part}; \text{meth. [19]}) &= 4823 \pm 77 \quad \text{for } \langle aGG \rangle \text{ value Eq. (18)} \qquad (26)\\ &= 4789 \pm 71 \quad \text{for } \langle aGG \rangle \text{ value Eq. (25)}\ . \qquad (27) \end{aligned}$$

The uncertainties $\pm 77, \pm 71$ in (26)–(27) are obtained by adding in quadrature the uncertainty from α_s (± 49), from m_s (± 24), from $\langle aGG \rangle$ ($\pm 49, \pm 37$), and from the resummation method uncertainty and $d_3^{(0)}$–uncertainty (± 25). The separate contributions to the central value 4823 in (26) are: 4079 from the leading term; 787 from the (resummed) canonical part (6); -54 from the $d=2$ strange mass term (7); 37 [2 if using (25)] from the $d=4$ gluon condensate term (8); -25 from the $d=4$ quark condensate terms (9); -0.5 from the $d=4$ quark mass terms (10).

We note that the uncertainties $\pm 77, \pm 71$ as obtained by us in (26)–(27) are lower than the uncertainty ± 113.2 for that quantity obtained by the authors of Ref. [8] (for $C_1 = 0.007\ \text{GeV}^{-2}$). The main reason for this is that we used different, in our opinion more updated, values of $\alpha_s^{\overline{\text{MS}}}(m_\tau^2)$ and $m_s(m_\tau^2)$ (23)–(24) than the values (19)–(20) used by Ref. [8]. The uncertainties $\pm 77, \pm 71$ are about the same as the uncertainties obtained in Ref. [8] for the "data" part, i.e., the first integral in (4), for $C_1 = 0.000$–$0.007\ \text{GeV}^{-2}$ (see Table. 1 of [8]). The authors of Ref. [8] chose $C_1 \approx 0.001$–$0.002\ \text{GeV}^{-2}$, i.e., their "theory" contribution was very small in comparison to their "data" contribution. The argument for the virtual exclusion of the theory from their considerations was that the uncertainties from their "theory" part were too high when C_1 is appreciable. We believe that a different approach, which emphasizes the "theory" part more than the "data" part, is legitimate as well, especially because the uncertainties from our analysis of the "theory" part are reasonable and comparable to the uncertainties of the "data" part even for large values of C_1. The authors of Ref. [8] obtained the values of the "data" parts for $C_1 \leq 0.007\ \text{GeV}^{-2}$, by using the available e^+e^- and τ decay data. Therefore, we choose the largest $C_1 = 0.007\ \text{GeV}^{-2}$ listed in their Table 1.

The "data" part given in their Table 1, for $C_1 = 0.007\ \text{GeV}^{-2}$, is

$$10^{11} \times a_\mu^{(\text{v.p.})}(s \leq s_0; \text{data part}) = 1622 \pm 56\ . \tag{28}$$

It corresponds to the first integral in Eq. (4).

Our evaluation of the theoretical part (26)–(27) of $s \leq s_0$ contributions excludes any QED corrections to the hadronic vacuum polarization. The leading term of such QED corrections corresponds to the exchange of a virtual photon within the hadronic blob in Fig. 2. This diagram induces in $R_{e^+e^-}$ in the master formula (1) the real photon emissions and the virtual (plus soft photon emissions) corrections to the pure hadronic final states.

Since the theory part in Eq. (4) (2nd line) is dominated by the perturbative QCD (pQCD) Adler function, this QED correction can be easily implemented by replacing $(1+D_{\rm can.}(Q^2))$ in (5) with the QED corrected one:

$$\left(1 + D_{\rm can.}(Q^2)\right)\left(1 + \frac{3Q_f^2}{4\pi}\alpha_{\rm em}(Q^2)\right) . \tag{29}$$

The QED corrections to the power-suppressed terms in (5) are negligibly small. Then, since the running of $\alpha_{\rm em}(Q^2)$ along the integration contour can be safely ignored, the QED correction can be seen to shift the theory part by the factor:

$$\left[1 + \frac{3}{4\pi}(\sum Q_f^4)/(\sum Q_f^2)\alpha_{\rm em}(s_0)\right] = 1 + \alpha_{\rm em}(s_0)/4\pi \approx 1 + 6 \times 10^{-4} \tag{30}$$

Thus, the QED correction to the theory part of $a_\mu^{\rm (v.p)}$ is approximately $(6 \times 10^{-4}) \times 4800 \approx 3$. It is small.

On the other hand, in the data part (28), the aforementioned class of QED contributions is, in principle, already included. This data part is taken from Ref. [8], and corresponds to the first integral in Eq. (4), for $C_1 = 0.007$ GeV^{-2}, with $R_{e^+e^-}(s)$ there based largely on the experimental e^+e^- and τ–decay data. In this connection, we mention that there is some controversy on whether the specific photon decay channels via ρ meson $[e^+e^- \to (\rho) \to \pi\pi\gamma]$ are included in the data or should be added to them. The contribution of these channels to the data part (28), in the narrow width approximation, would be about (18 ± 4). However, the τ–decay data already include this decay channel, while the situation with the e^+e^- data in this respect is less clear. Since the authors of Ref. [8] included in their analysis both types of data, we presume that the additional possible contributions from the aforementioned photonic decay channels to the data part (28) are significantly lower than (18 ± 4), and we will therefore ignore such contributions.

We note that the inclusive, total QED correction to $a_\mu^{\rm (v.p.)}$ is very small when compared to the contribution from the real photon emissions. From the above QED correction to the theory part we can estimate the total QED correction to $a_\mu^{\rm (v.p.)}$ as $a_\mu^{\rm (v.p.)} \times \alpha_{\rm em}(s_0)/4\pi \approx 7000 \times (6 \times 10^{-4}) \approx 4$. This is much smaller than the contribution from the real photon emissions, which is according to [23] about 90 units from the photonic decay channels of mesons, e.g. ρ, ω, and ϕ. These two contributions may appear incompatible, but there is no real contradiction. It can be seen most easily in the narrow width approximation (n.w.a.) of the resonances. The n.w.a. formulas for the portions of R_{e+e-} in (1) from the various decay channels of a resonance contain as factors the corresponding branching ratios, whose sum is always one. Thus, an increase in the photonic decay channel contributions must be compensated by the decrease in the pure hadronic channel contributions by exactly the same amount. The contributing QED correction factor in the Adler function in Eq. (29) thus accounts for the QED corrections in the continuum only. Beyond the n.w.a. approximation, however, the cancellation between the photonic channel contributions and the pure hadronic channel contributions cannot be exact, since the decay widths of the resonances are sizable, and so a small QED correction survives after the cancellation. This remaining QED correction, along with the QED corrections in the continuum, are those accounted for by the QED correction factor in the Adler function in Eq. (29).

We now include in our theory–part contributions (26)–(27) the aforementioned QED contributions (29).

$$10^{11} \times a_\mu^{(\mathrm{v.p.})}(s \le s_0; \mathrm{th.\ part}; \mathrm{meth.\ [19]}) = 4826 \pm 77 \quad \text{for } \langle aGG \rangle \text{ value Eq. (18)} \tag{31}$$
$$= 4792 \pm 71 \quad \text{for } \langle aGG \rangle \text{ value Eq. (25)} . \tag{32}$$

This implies for the sum of the "theory" (31)–(32) and the "data" (28) parts

$$10^{11} \times a_\mu^{(\mathrm{v.p.})}(s \le s_0) = 6448 \pm 95 \quad \text{for } \langle aGG \rangle \text{ value Eq. (18)} \tag{33}$$
$$= 6414 \pm 90 \quad \text{for } \langle aGG \rangle \text{ value Eq. (25)} . \tag{34}$$

The authors of Ref. [8] obtained for this quantity the value 6343 ± 60, by almost excluding their "theory" part (complete exclusion of the "theory" part gave them 6350.5 ± 74).

If we decreased parameter C_1 from 0.007 GeV^{-2} toward zero, our central values in (33)–(34) would go down toward the pure "data" value 6351, i.e., by ± 97 and ± 63 units, respectively. We will comment on this point in the next Section.

If we add to the obtained quantities (33)–(34) the hadronic vacuum polarization parts from the region $s > s_0$ as obtained in Ref. [8],[5] we obtain

$$10^{11} \times a_\mu^{(\mathrm{v.p.})} = 7029 \pm 96 \quad \text{for } \langle aGG \rangle \text{ value Eq. (18)} \tag{35}$$
$$= 6995 \pm 91 \quad \text{for } \langle aGG \rangle \text{ value Eq. (25)} . \tag{36}$$

The result obtained in the analysis of Ref. [8] is

$$10^{11} \times a_\mu^{(\mathrm{v.p.})} = 6924 \pm 62 \quad \text{(D.H. [8])} . \tag{37}$$

While the bulk of the result (37) of Ref. [8] was obtained by taking into account the data on e^+e^- and the τ decays, the bulk of the result (35)–(36) is obtained here by careful resummation of the contour integral of (4) where we account for the renormalon structure of the associated Adler function and for the $d > 0$ terms.

IV. COMPARISONS

How do our results compare with recent results of others on $a_\mu^{(\mathrm{v.p.})}$?

---

[5] They obtained 581 ± 15 for this contribution, using for the pQCD parts apparently the value (19) for α_s. The authors of Ref. [23] obtained slightly higher values 584 ± 9, once we subtract from their values the (pQCD) contributions from (3 GeV$^2 < s < s_0$) ($s_0 = 1.8^2$ GeV^2). They used lower values for α_s: $\alpha_s(m_\tau^2) \approx 0.3088 \pm 0.0245$. This small discrepancy would apparently increase once we adjusted α_s to the same value, say (19), for the two approaches of Refs. [8,23]). Then the prediction of Ref. [23] would be about 592 ± 10. We decide to take the value of Ref. [8]: 581 ± 15. We can reproduce their pQCD–parts with the simple TPS approach for $R_{ee}(s)$. When we use, instead, the approaches that account for the renormalon structure of $R_{ee}(s)$, and/or we use the value (23) instead of (19) for $\alpha_s(m_\tau^2)$, the values of these contributions change insignificantly. QED corrections are insignificant.

In Table I we show the values of $a_\mu^{(\mathrm{v.p.})}$ as predicted by others [24–27,23,8], along with our values.[6] We see that our values are consistent with the values obtained by most of the other authors. We recall the fact that the bulk of our result is obtained by the pQCD+renormalon calculation, while the bulk of the results of the others was obtained by data integration. We observe that the consistency with the results of the others is even slightly stronger when we take the value of the gluon condensate of Eq. (25), i.e., the small gluon condensate value obtained by the ALEPH analysis [20] and suggested also by the renormalon formalism of [22]. Several of the entries in the Table are based on inclusion of the τ decay data.[7]

How do our results (35)–(36) compare with the experimental predictions for a_μ? This question remains somewhat unclear due to theoretical uncertainties of several higher order hadronic contributions, as argued by the authors of Ref. [23]. The largest theoretical uncertainty is in the calculation of the hadronic light–by–light (l.l.) contributions. The chiral model (ch.m.) approaches would predict $10^{11} \times a_\mu^{(\mathrm{l.l})} = -86 \pm 25$ [32]; the quark constituent model (q.c.m.) would predict $+92 \pm 20$ [23]. The quark constituent model is valid for large values of the virtual photon momenta only, so we will give results for the chiral model unless otherwise stated. The QED corrections of the type of Fig. 2 have already been included in (35)–(36). In addition, there are other QED radiative corrections where the photon propagator does not have both ends attached to the hadronic blob (the "rest") $10^{11} \times a_\mu(\mathrm{rad.corr., rest}) = -101 \pm 6$ [33]. These two radiative corrections (-86 ± 26) and (-101 ± 6) have to be added to the entries of Table I to obtain the total hadronic contribution $a_\mu^{(\mathrm{hadr.})}$. In our case, we add them to the hadronic vacuum polarization contribution (35)–(36) (we add the uncertainties in quadrature), and obtain

$$
\begin{aligned}
10^{11} \times a_\mu^{(\mathrm{hadr.})} &= 6842 \pm 100 \quad \text{for } \langle aGG \rangle \text{ value Eq. (18)} \qquad (38)\\
&= 6808 \pm 95 \quad \text{for } \langle aGG \rangle \text{ value Eq. (25)}\ . \qquad (39)
\end{aligned}
$$

If we took, instead, the quark constituent model result for the light–by–light contributions, we would obtain 7014 ± 98 and 6980 ± 93, respectively.

We can now add the results (38)–(39) to the well known electroweak contribution ([34,35] and references therein)

$$
10^{11} \times a_\mu(\mathrm{EW}) = 116\ 584\ 858 \pm 5 \qquad (40)
$$

and obtain the predictions

$$
\begin{aligned}
10^{11} \times a_\mu(\mathrm{predicted}) &= 116\ 591\ 700 \pm 100 \quad \text{for } \langle aGG \rangle \text{ value Eq. (18)} \qquad (41)\\
&= 116\ 591\ 666 \pm 95 \quad \text{for } \langle aGG \rangle \text{ value Eq. (25)}\ . \qquad (42)
\end{aligned}
$$

The actual experimental number [1], when averaged with the older measurements [36], is

---

[6] We did not include in the Table the results of some earlier analyses [28].

[7] The entries of Table I are based on the use of the dispersion relation (1). The latter has been questioned in Ref. [29], because there is no proof that the photon propagator is at least polynomially bounded. The inclusion of the τ decay data for calculation of $a_\mu^{(\mathrm{v.p.})}$ has been questioned in Ref. [30], and the difficulties connected with this inclusion have been investigated in Ref. [31].

$$10^{11} \times a_\mu(\text{experiment, averaged}) = 116\ 592\ 030 \pm 152\ . \tag{43}$$

The predictions (41), (42) thus differ from the experimental result by 330 ± 182 (1.81σ) and 364 ± 179 (2.03σ), respectively, where $\sigma = 182, 179$ is obtained by adding in quadrature the experimental ($\sigma_{\text{exp.}} = 152$) and the theoretical uncertainty ($\sigma_{\text{th.}} = 100, 95$).[8] If we took, instead of the chiral model result, the quark constituent model result for the light–by–light contributions, the corresponding deviations would be 152 ± 181 (0.84σ) and 186 ± 178 (1.04σ), respectively.

If we take only the newest experimental number [1]

$$10^{11} \times a_\mu(\text{experiment, new}) = 116\ 592\ 020 \pm 152\ , \tag{44}$$

then the predictions (41)–(42) differ from it by 320 ± 182 (1.76σ) and 354 ± 179 (1.98σ), respectively. These deviations would be 0.78σ and 0.99σ, respectively, if we used the quark constituent model results for the light–by–light contributions.

V. SUMMARY

We obtain clear deviations of the theoretical results from the experimental ones when we use the chiral approach results for the light–by–light contributions. However, these deviations nonetheless are significantly smaller than the 430 ± 165 (2.6σ) difference [1][9] that has been suggested as the appearance of new physics. The deviations of our predictions from the new experimental values (43), (44), range from 320 ± 182 (1.76σ) to 364 ± 179 (2.03σ), depending mostly on the taken values of the gluon condensate. Our results are consistent with those of most of the other authors who used data integration for the bulk of their results. In contrast, the bulk of our results was obtained by pQCD+renormalon calculation.

A major contribution to the aforementioned reduction of the deviations are our values for the hadronic vacuum polarization contributions (CLS1 and CLS2 in Table I), which are significantly higher than those of Davier and Höcker [8] (DH [8] in Table I). Our values originate from resummation of a major part of this contribution via a contour integral in the complex energy plane and accounting for the renormalon structure of the associated Adler function in the integrand. The accounting for the renormalon structure contributes about 50 units to $10^{11} \times a_\mu^{(\text{v.p.})}$, and this is larger than the expected future experimental uncertainties.

---

[8] The decrease of C_1 to zero would contribute, in quadrature, additional 97 and 63 units to the uncertainties, respectively – as mentioned in the previous Section. In such a case, the predictions would deviate from the experimental result by 330 ± 206 and 364 ± 190, respectively.

[9] There, $\sigma \approx 165$ was obtained by adding in quadrature $\sigma_{\text{exp.}} \approx 150$ and $\sigma_{\text{th.}} \approx 67$, where the value of $\sigma_{\text{th.}}$ was taken from Ref. [8].

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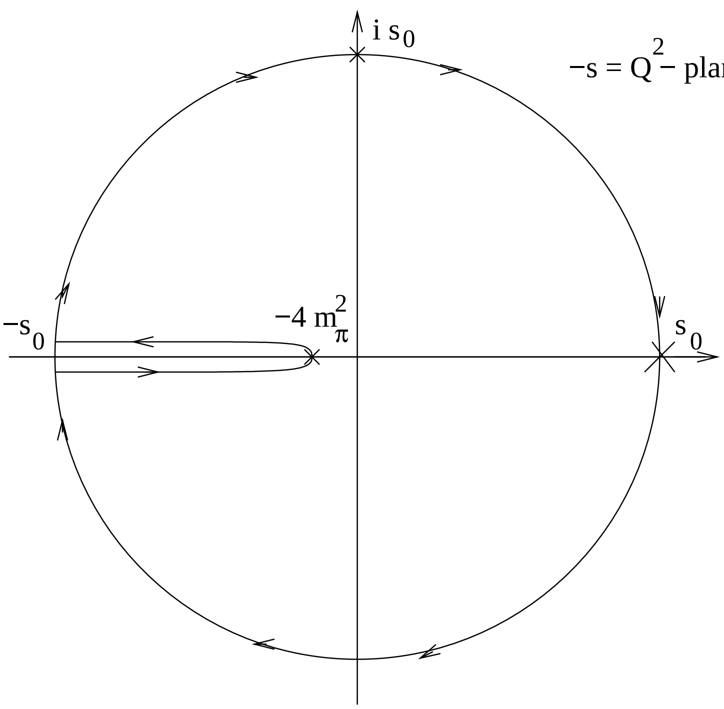

FIG. 1. Application of the Cauchy theorem to the depicted integration contour in the Q^2–plane leads to the contour integral in Eq. (4).

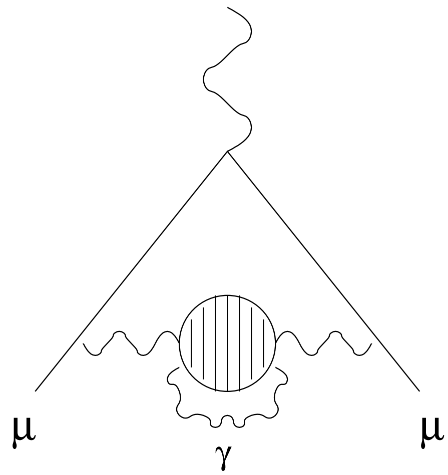

FIG. 2. QED correction to the hadronic blob.

authors	$10^{11} \times a_\mu^{(\text{v.p.})}$	method
ADH [24]	7011 ± 94	$e^+e^- + \tau$ data
BW [25]	7026 ± 160	e^+e^- data
J [26]	6974 ± 105	mostly e^+e^- data
N1 [27]	7031 ± 77	$e^+e^- + \tau$ data
N2 [27]	7011 ± 117	e^+e^- data
TY1 [23]	7002 ± 65	$e^+e^- + \tau + \text{spacel.} F_\pi(t)$ data
TY2 [23]	6982 ± 97	$e^+e^- + \text{spacel.} F_\pi(t)$ data
DH [8]	6924 ± 62	mostly $e^+e^- + \tau$ data
CLS1	7029 ± 96	mostly theory, and (18) value
CLS2	6995 ± 91	mostly theory, and (25) value

TABLE I. Comparison of predictions of the hadronic vacuum polarization contributions to a_μ by various authors.